GENERALIZED SELF-CONSISTENT-FIELD METHOD FOR DETERMINING THE ELASTIC PROPERTIES OF HETEROGENEOUS MATERIALS

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The elastic characteristics of heterogeneous materials have been the subject of numerous experimental and theoretical studies for many years. This is not surprising, since they are among the most important characteristics of structural (especially composite) materials and to a large extent determine their service properties.

However, it should be noted that many of the phenomenological and theoretical methods of evaluating the elastic characteristics of heterogeneous materials from their structural features and properties of the constituent phases are often based on formal assumptions of a mathematical nature which facilitate the solution of equations that take into account the complex character of interaction of the strutural elements of the material. In this case, there is no clear relationship between the simplifying assumptions used and the corresponding changes in the physical model of the heterogeneous system. Nonetheless, different assumptions sometimes lead to the same results. This is obviously fairly unambiguous evidence of the equivalence of the physical models — an equivalence which is not always taken into consideration in the choice of the theoretical relations used to study the elastic characteristics of actual heterogeneous materials with a statistical and matrix structure.

We will show that many of the best-known results can be obtained from the solution of the Eshelby problem [1] on the deformation of an elastic exogeneous inclusion in an infinite uniform matrix. Let $\tilde{\varepsilon}$ be the uniform strain of a medium whose properties are characterized by the elastic moduli tensor $\mathbf{c}^m$. Then in accordance with the Eshelby solution, the strain of a single inclusion

$$\varepsilon^f = \{\mathbf{I} + \mathbf{N} : [(\mathbf{c}^m)^{-1} : \mathbf{c}^f - \mathbf{I}]\}^{-1} : \tilde{\varepsilon}. \quad (1)$$

Here, $\mathbf{N}$ is the Eshelby tensor; $\mathbf{c}^f$ is the tensor of the elastic moduli of the inclusion; $\mathbf{I}$ is the unit tensor.

If the material of the matrix and inclusion is isotropic and if the form of the latter is spherical, then the tensor $\mathbf{N}$ will also be isotropic. In this case, we can use the representation [2]

$$(\mathbf{c}^m)^{-1} = \frac{1}{3K^m}\mathbf{V} + \frac{1}{2\mu^m}\mathbf{D}, \quad \mathbf{c}^f = 3K^f\mathbf{V} + 2\mu^f\mathbf{D},$$
$$\mathbf{N} = \frac{3K^m}{3K^m + 4\mu^m}\mathbf{V} + \frac{6(K^m + 2\mu^m)}{5(3K^m + 4\mu^m)}\mathbf{D},$$

where $K^{m(f)}$, $\mu^{m(f)}$ are the bulk and shear moduli of the matrix and inclusion; $\mathbf{V}$ and $\mathbf{D}$ are the volumetric and deviatoric components of the unit tensor ($\mathbf{I} = \mathbf{V} + \mathbf{D}$).

Using such a representation for the tensors in conjunction with Eq. (1), we can write the following for the strain of an exogeneous inclusion placed in an infinite medium

$$\varepsilon^f = \left[\frac{\mathbf{V}}{1 + a(K^f - K^m)} + \frac{\mathbf{D}}{1 + b(\mu^f - \mu^m)}\right] : \tilde{\varepsilon}$$
$$\left(a = \frac{3}{3K^m + 4\mu^m}, \quad b = \frac{6(K^m + 2\mu^m)}{5\mu^m(3K^m + 4\mu^m)}\right). \quad (2)$$

Now we use Eq. (2) to determine the effective elastic characteristics of the two-phase heterogeneous system. Let the elastic characteristics of the first phase be determined by the isotropic tensor $\mathbf{c}^{(1)}$, and let the characteristics of the second be determined by the tensor $\mathbf{c}^{(2)}$.

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To establish the effective properties of these systems, we will examine a uniform comparison body with the elastic characteristics K_c and μ_c. The comparison body is distinguished by the fact that the elastic field in the single spherical inclusion — with the elastic characteristics of a separate phase component — is an average over the volume occupied by the given phase in the heterogeneous material. Here, of course, we assume that the strain of the comparison body and the macrostrain of the actual system coincide, i.e., on the average the elastic behavior of the phase components is the same as in the corresponding isolated inclusions placed alternately in the homogeneous medium. Then, in accordance with Eq. (2), we can write the following for the mean strain tensors in the first and second phases

$$\boldsymbol{\varepsilon}_i=\left[\frac{\mathbf{V}}{1+a_c(K_i-K_c)}+\frac{\mathbf{D}}{1+b_c(\mu_i-\mu_c)}\right]:\langle\boldsymbol{\varepsilon}\rangle,\quad i=1,\ 2. \tag{3}$$

Here, as previously [see Eq. (2)],

$$a_c=\frac{3}{3K_c+4\mu_c},\quad b_c=\frac{6(K_c+2\mu_c)}{5\mu_c(3K_c+4\mu_c)}. \tag{4}$$

For the mean average deformation in heterogeneous systems, also, it holds for the equation

$$\langle\boldsymbol{\varepsilon}\rangle=c_1\boldsymbol{\varepsilon}_1+c_2\boldsymbol{\varepsilon}_2 \tag{5}$$

(c_1 and c_2 are the volume contents of the materials of the first and second phases in the heterogeneous material). Inserting Eq. (3) into Eq. (5), we have

$$\begin{gathered}\frac{c_1}{1+a_c(K_1-K_c)}+\frac{c_2}{1+a_c(K_2-K_c)}=1,\\ \frac{c_1}{1+b_c(\mu_1-\mu_c)}+\frac{c_2}{1+b_c(\mu_2-\mu_c)}=1.\end{gathered} \tag{6}$$

The effective elastic characteristics of the heterogeneous two-phase material ($\mathbf{c}^*$) are easily found from the condition

$$\langle\boldsymbol{\sigma}\rangle=c_1\boldsymbol{\sigma}_1+c_2\boldsymbol{\sigma}_2, \tag{7}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the tensors of the mean stresses over the volumes of the heterogeneous material occupied by the first and second phases, respectively. With satisfaction of the generalized Hooke's law, Eq. (7) can be rewritten in the following equivalent form:

$$\mathbf{c}^*:\langle\boldsymbol{\varepsilon}\rangle=\mathbf{c}^{(1)}:\boldsymbol{\varepsilon}_1c_1+\mathbf{c}^{(2)}:\boldsymbol{\varepsilon}_2c_2. \tag{8}$$

With allowance for the expansions $\mathbf{c}^{(1)}=3K_1\mathbf{V}+2\mu_1\mathbf{D}$, $\mathbf{c}^{(2)}=3K_2\mathbf{V}+2\mu_2\mathbf{D}$, $\mathbf{c}^*=3K^*\mathbf{V}+2\mu^*\mathbf{D}$ and Eq. (3), we can use Eq. (8) to find

$$\begin{gathered}K^*=\frac{K_1c_1}{1+a_c(K_1-K_c)}+\frac{K_2c_2}{1+a_c(K_2-K_c)},\\ \mu^*=\frac{\mu_1c_1}{1+b_c(\mu_1-\mu_c)}+\frac{\mu_2c_2}{1+b_c(\mu_2-\mu_c)}.\end{gathered} \tag{9}$$

The effective bulk modulus K^* and shear modulus μ^* completely characterize the elastic properties of the isotropic heterogeneous material.

Performing identical transformations in Eqs. (9) and allowing for (4) and (6), we find expressions for the bulk and shear moduli:

$$\begin{gathered}K^*=\langle K\rangle-\frac{c_1c_2(K_1-K_2)^2}{\frac{4}{3}\mu_c+c_1K_2+c_2K_1},\\ \mu^*=\langle\mu\rangle-\frac{c_1c_2(\mu_1-\mu_2)^2}{\frac{\mu_c(9K_c+8\mu_c)}{6(K_c+2\mu_c)}+c_1\mu_2+c_2\mu_1}.\end{gathered} \tag{10}$$

Here, $\langle K\rangle$ and $\langle\mu\rangle$ are the mean values of the bulk and shear elastic moduli of the heterogeneous material: $\langle K\rangle = c_1K_1 + c_2K_2$, $\langle\mu\rangle = c_1\mu_1 + c_2\mu_2$.

Equations (10) coincide with the formulas of the generalized singular approximation in [2]. These formulas can be used to obtain many of the well-known solutions if the characteristics of the comparison body are regarded as variables. Here, with a change in the characteristics of the comparison body, the solutions that are obtained are reduced to a form

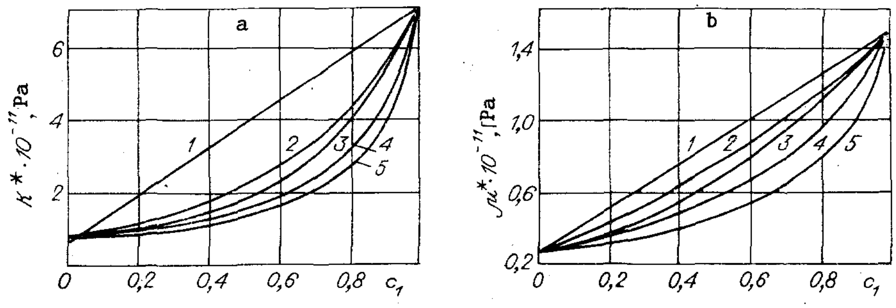

Fig. 1

corresponding to different physical models of the heterogeneous material. For example, if we put $K_c = K^*$ and $\mu_c = \mu^*$, we arrive at the self-consistent solution in [3] for symmetrical statistical systems with interpenetrating components. The solution proposed above is nothing more than a generalization of this approach due to the simultaneous use of additivity equations for the averaged fields of the microstrains and microstresses. Also, as was shown in [2], if we assume that $K_c = 0$, $\mu_c = 0$, and $K_c = \infty$, $\mu_c = \infty$, we obtain the lower and upper bounds of the effective characteristics corresponding to the simplest models of a heterogeneous material with a uniform stress and strain distribution:

$$\langle K^{-1}\rangle^{-1} \leqslant K^* \leqslant \langle K\rangle, \langle\mu^{-1}\rangle^{-1} \leqslant \mu^* \leqslant \langle\mu\rangle.$$

If we then put $K_c = K_1$, $\mu_c = \mu_1$ and $K_c = K_2$, $\mu_c = \mu_2$, we find the Hashin–Strickmen variational principles [4]:

$$K_1 + \frac{c_2(K_2 - K_1)}{1 + c_1 a_1 (K_2 - K_1)} \leqslant K^* \leqslant K_2 + \frac{c_1(K_1 - K_2)}{1 + c_2 a_2 (K_1 - K_2)},$$

$$\mu_1 + \frac{c_2(\mu_2 - \mu_1)}{1 + c_1 b_1 (\mu_2 - \mu_1)} \leqslant \mu^* \leqslant \mu_2 + \frac{c_1(\mu_1 - \mu_2)}{1 + c_2 b_2 (\mu_1 - \mu_2)},$$

where $K_2 > K_1$ and $\mu_2 > \mu_1$.

As was noted in [2], these equations – corresponding to boundary values of the elastic characteristics – can be used to find the effective elastic characteristics of matrix systems.

Finally, with $K_c = \langle K\rangle$ and $\mu_c = \langle\mu\rangle$, Eqs. (10) become expressions corresponding to those obtained within the framework of the methods of singular approximation [2], strong isotropy [5], conditional moments [6], and limiting locality [7]. This once again underlines the equivalence of the physical models corresponding to different mathematical solutions.

An illustration of the use of the above relations is given by the results, shown in Fig. 1, of calculation of the concentration dependences of the bulk (a) and shear (b) moduli of a W–Al composite with different comparison-body characteristics: 1) $K_c = \infty$, $\mu_c = \infty$; 2) $K_c = K_1$, $\mu_c = \mu_1$; 3) $K_c = K^*$, $\mu_c = \mu^*$; 4) $K_c = K_2$, $\mu_c = \mu_2$; 5) $K_c = 0$, $\mu_c = 0$. It can be seen from the figure that the paths of the concentration curves differ appreciably, depending on the choice of physical model for the material (which is determined by the assignment of the corresponding characteristics of the comparison body). Here, we should point out the dependence of the elastic properties on the volume content of tungsten in the composite (c_1) found for the model in the form of a statistical system of geometrically equal phase components ($K_c = K^*$, $\mu_c = \mu^*$). In the case of a low volume concentration of tungsten, when this equality is not manifest, the calculated values are close to the corresponding values for a system with an aluminum matrix (curve 4). Since tungsten obviously becomes the matrix phase at $c_1 \to 1$, this is reflected on the graph by the convergence of curves 2 and 3.

In the case of heterogeneous materials containing inclusions of ellipsoidal or cylindrical form, the scheme used to calculate the effective elastic characteristics remains the same. However, in this case the Eshelby tensor will not be characterized by two constants, as was the tensor of the effective elastic characteristics of the heterogeneous system examined above. Thus, for a fiber composite, the matrix of the elastic moduli will contain five unknown constants. In accordance with Eqs. (2) and (8), the tensor of the effective elastic moduli is found from the relation

$$\mathbf{c}^* = \sum c_i \mathbf{c}^i : \{\mathbf{I} + \mathbf{N} : [(\mathbf{c}^c)^{-1} : \mathbf{c}^{(i)} - \mathbf{I}]\}^{-1}.$$

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NUMERICAL INVESTIGATION OF RECIRCULATION FLOWS IN A THREE-DIMENSIONAL CAVERN

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The problem of viscous incompressible fluid flow in a three-dimensional cavity initiated by a moving upper lid is considered. The numerical solution of the Navier–Stokes equations is sought on a grid with diversity velocities in the vector potential-vortex variables. New structures corner vortices and Taylor-Görtler type vortices inherent to three-dimensional flows are obtained numerically. The dependence of the flow nature on the Reynolds number Re and on the ratio between the cavity width to its depth is investigated.

In a number of cases spatial effects can substantially influence the incompressible fluid flow pattern. Consequently solutions obtained when using two-dimensional approximations differ significantly from the experimental data. A typical example is the problem of viscous incompressible fluid flow in a three-dimensional cavity with a moving upper lid. Application of the two-dimensional Navier–Stokes equations assumes that the cavity width L (Fig. 1) is much greater than its depth H. The ratio of the width to the depth of channels varied between 1 and 3 in known experiments [1, 2]. The presence of endface walls and the boundedness of the channel width cause considerable flow reconstruction as compared with the plane case. Numerical computations of viscous fluid flow in a cubic cavern are performed in [3, 4] by using pseudospectral and implicit multigrid methods.

FORMULATION OF FLUID FLOW PROBLEMS IN TWO- AND THREE-DIMENSIONAL CHANNELS WITH A MOVING LID

The problem of two-dimensional fluid flow in a cavity of rectangular section with a moving lid is typical for testing different numerical algorithms [5, 6]. A viscous incompressible fluid flow is examined in a rectangular domain of length B and height H. The fluid is at rest at the initial time, and the upper lid is set in motion at a constant velocity u_0. Adhsesion conditions are given on the cavern boundaries. It is required to determine the stationary laminar flow pattern as a function of Re.

The problem is the following for flows in a three-dimensional cavern. The solution is sought in a domain D (Fig. 1)

$$D = \{(x,\ y,\ z)\colon 0 \leqslant x \leqslant B,\ 0 \leqslant y \leqslant H,\ 0 \leqslant z \leqslant L\}.$$

The moving lid (y = 0) moves from right to left. The boundary conditions are: $u(x, 0, z) = 1$, $v(x, 0, z) = w(x, 0, z) = 0$ for $y = 0$; the velocity vector components u, v, w equal zero on the remaining boundaries. The initial conditions are selected either as at rest ($u = v = w = 0$) or values of the desired parameters are used for a certain smaller Re.

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